THE CONTACT PROBLEM OF A PLATE ON AN ELASTIC FOUNDATION

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This paper considers the equilibrium of an unbounded elastic layer lying on a rigid immovable base and deformed by the action of an elastic, symmetrically loaded, circular plate which is in contact with it over its entire surface (Fig. 1).

Absence of friction between the plate and the layer as well as between the layer and the base is assumed. The plate will be assumed to have unit radius, the thickness of the layer will be denoted by h and that of the plate by δ.

We will introduce into consideration the pressure $p(r)$ in the region of contact between the plate and the layer. This makes it possible to reduce the problem to the consideration of two problems. The first will involve the study of the equilibrium of the layer under the effect of the symmetric loading $p(r)$ distributed over the contact region (Fig. 2), the second comprises the study of the equilibrium of the plate under the action of a system of external loads and the pressure $p(r)$ (Fig. 3).

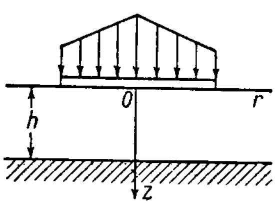

FIG. 1.

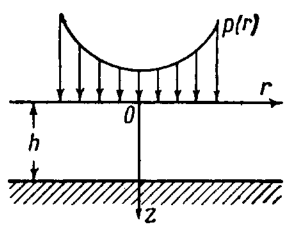

FIG. 2.

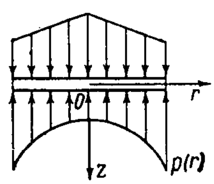

FIG. 3.

1. We will find the link between the pressure and the normal displacements of the surface of the layer in the region of contact. For this

purpose we will follow [1].

For the above assumptions the problem reduces to the integration of the equations of elasticity in cylindrical coordinates r, ϕ, z with the boundary conditions

$$\tau_{rz} = 0, \quad w = C \quad \text{for } z = h, \qquad \tau_{rz} = 0 \quad \text{for } z = 0 \tag{1.1}$$

$$\sigma_z = -p(z) \quad (r < 1), \qquad \sigma_z = 0 \quad (r > 1) \quad \text{for } z = 0 \tag{1.2}$$

Here, and later, u, v, w are the components of the stress tensor in the described cylindrical coordinate system and σ_z, τ_{rz} are the normal and tangential stresses.

As a consequence of the symmetry of the problem, $v = 0$ and the remaining quantities do not depend on the coordinate ϕ. We will use the representation of the displacement vector in terms of harmonic functions due to Papkovich-Neuber

$$2\mu u = -\frac{\partial\Phi}{\partial r}, \qquad 2\mu w = -\frac{\partial\Phi}{\partial r} + 4(1-\nu)\Phi_1, \qquad \Phi = \Phi_0 - z\Phi_1 \tag{1.3}$$

Here μ is the shear modulus, ν Poisson's ratio of the material of the layer and Φ_0, Φ_1 are functions which are harmonic in the region of the layer. The stresses σ_z and τ_{rz} are expressed in terms of these functions by the formulas

$$\sigma_z = 2(1-\nu)\frac{\partial\Phi_1}{\partial z} - \frac{\partial\Phi_2}{\partial z} - z\frac{\partial^2\Phi_1}{\partial z^2}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-2\nu)\Phi_1 - \Phi_2 - z\frac{\partial\Phi_1}{\partial z}\right], \qquad \Phi_2 = \frac{\partial\Phi_0}{\partial z} \tag{1.4}$$

It has been shown in [1] that by taking the harmonic functions Φ_1, Φ_2 in the integral representations

$$\Phi_1 = \int_0^\infty A(\lambda)\,\mathrm{sh}\,\lambda(h-z)\,J_0(\lambda r)\,\frac{d\lambda}{\mathrm{sh}\,\lambda h}$$

$$\Phi_2 = (1-2\nu)\Phi_1 + \int_0^\infty \lambda h A(\lambda)\,\frac{\mathrm{sh}\,\lambda z}{\mathrm{sh}^2\lambda h}\,J_0(\lambda r)\,d\lambda \tag{1.5}$$

we satisfy the boundary conditions (1.1). The function $A(\lambda)$ is as yet unknown and $J_0(y)$ is the Bessel function. We will express the displacement w of the upper boundary of the layer and the normal stresses on it in terms of the function $A(\lambda)$

$$w = \vartheta\int_0^\infty A(\lambda)\,J_0(\lambda r)\,d\lambda, \qquad \sigma_z = -\int_0^\infty \lambda A(\lambda)\,\frac{J_0(\lambda r)}{1-g(\lambda)}\,d\lambda \tag{1.6}$$

Here ϑ is a coefficient characterizing the material of the layer

$$\vartheta = \frac{1-\nu}{\mu} \tag{1.7}$$

and $g(\lambda)$ is the following function of λ:

$$g(\lambda) = \frac{\lambda h + \operatorname{sh} \lambda h e^{-\lambda h}}{\lambda h + \operatorname{sh} \lambda h \operatorname{ch} \lambda h} \tag{1.8}$$

Substituting (1.6) into (1.2) we find the conditions which must be imposed on the function $A(\lambda)$

$$p(r) = \int_0^\infty \frac{\lambda A(\lambda)}{1-g(\lambda)} J_0(\lambda r)\, d\lambda \quad (r < 1), \qquad \int_0^\infty \frac{\lambda A(\lambda)}{1-g(\lambda)} J_0(\lambda r)\, d\lambda = 0 \quad (r > 1) \tag{1.9}$$

2. We will now consider the equilibrium of the plate shown in Fig. 3. We will assume that the center of the plate has a given displacement W_0 (in certain cases it is convenient to take instead of W_0 the deflection of the plate at the edge). We denote by $V(r)$ the deflection of the plate, fixed at the center, under the action of all forces applied to it, in addition to the pressure $p(r)$ acting on the area of contact with the layer. The deflection under the action of the pressure $p(r)$ we denote by W. For the determination of the last we have the following equation and boundary conditions, corresponding to a free edge [2]:

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right)\right]\right\} = -\frac{p(r)}{D} \qquad \left(D = \frac{E\delta^3}{12(1-\nu^2)}\right) \tag{2.1}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right)\right] = 0 \qquad \text{for } r = 1 \tag{2.2}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right) - \frac{(1-\nu)}{r}\frac{dW}{dr} = 0 \qquad \text{for } r = 1 \tag{2.3}$$

$$W = 0 \qquad \text{for } r = 0 \tag{2.4}$$

Here D is the cylindrical rigidity of the plate, E Young's modulus of its material and ν Poisson's ratio. Using the expression p from (1.9), we integrate Equation (2.1) to give

$$W = -\frac{1}{D}\int_0^\infty \frac{A(\lambda)}{1-g(\lambda)} \left[J_0(\lambda r) + C_1(\lambda)\frac{r^2}{4} + \right. \tag{2.5}$$

$$\left. + C_2(\lambda)\frac{r^2}{4}(\ln r - 1) + C_3(\lambda)\ln r + C_4(\lambda)\right] d\lambda$$

As a consequence of the boundedness of the displacement W at the point $r = 0$ one must set $C_3(\lambda) = 0$. Satisfying the conditions (2.2), (2.3) and (2.4) we find

$$C_1(\lambda) = \lambda^2 J_0(\lambda) + \frac{1-\nu}{1+\nu}\Big[-\lambda^2 J_2(\lambda) + \frac{\lambda^3}{2} J_1(\lambda)\Big]$$
$$C_2(\lambda) = -\lambda^3 J_1(\lambda), \qquad C_4(\lambda) = -1 \qquad (2.6$$

The total displacement of the points of the plate in correspondence with what has been said at the beginning of the section is

$$w = W_0 + V(r) + W(r) \qquad (2.7)$$

Equating the displacement of the plate to the vertical displacement of the upper boundary of the layer in the region of contact we obtain the second integral condition which must be imposed on the function $A(\lambda)$

$$\vartheta \int_0^\infty \frac{A(\lambda)}{1-g(\lambda)} \Big\{ J_0(\lambda r)(1-g) + \qquad (2.8)$$
$$+ \frac{1}{\vartheta D \lambda^3}\Big[J_0(\lambda r) + C_1 \frac{r^2}{4} + C_2 \frac{r^2}{4}(\ln r - 1) - 1\Big]\Big\} d\lambda = W_0 + V(r) \quad (r < 1)$$

Equations (1.9) and (2.8) represent a system of two integral equations for the determination of the function $A(\lambda)$.

3. Following [1] we will seek the solution of the system $A(\lambda)$ in the form

$$\frac{\vartheta A(\lambda)}{1-g(\lambda)} = \int_0^1 \varphi(t) \cos \lambda t \, dt \qquad (3.1)$$

where $\phi(t)$ is a new unknown function.

We will carry out on the right-hand side of (3.1) an integration by parts and substitute the result into the second equality (1.9)

$$\int_0^\infty \frac{\lambda A(\lambda)}{1-g} J_0(\lambda r) \, d\lambda = \frac{1}{\vartheta}\Big[\varphi(1) \int_0^\infty J_0(\lambda r) \sin \lambda \, d\lambda - \int_0^1 \varphi'(t) \, dt \int_0^\infty J_0(\lambda r) \sin \lambda t d\lambda\Big]$$

Using the known formula

$$\int_0^\infty J_0(\lambda r) \sin \lambda t \, d\lambda = \begin{cases} 0 & \text{при } 0 \leqslant t < r \\ (t^2 - r^2)^{-1/2} & \text{при } t > r \end{cases} \qquad (3.2)$$

we verify that the integrals on the right-hand side vanish for $r > 1$; therefore, the second condition (1.9) is satisfied identically.

Substitution of $A(\lambda)$ from (3.1) into (2.8), change of order of integration and utilization of the formula

$$\int_0^\infty J_0(\lambda r)\cos\lambda t d\lambda = \begin{cases} 0 & \text{for } t > r \\ (r^2 - t^2)^{-1/2} & \text{for } 0 \leqslant t < r \end{cases}$$

leads to the relation $(r < 1)$

$$\int_0^r \frac{\varphi(t)\,dt}{\sqrt{r^2 - t^2}} + \int_0^1 \varphi(t) \int_0^\infty \Big\{-g(\lambda)J_0(\lambda r) + \tag{3.3}$$

$$+\frac{1}{\vartheta D\lambda^3}\Big[J_0(\lambda r) - 1 + C_1\frac{r^2}{4} + C_2\frac{r^2}{4}(\ln r - 1)\Big]\Big\}\cos\lambda t\, d\lambda\, dt = W_0 + V(r)$$

We will use for the transformation of the second integral the formulas

$$J_0(\lambda r) = \frac{2}{\pi}\int_0^{1/2\pi}\cos(\lambda r\sin\theta)\,d\theta, \quad r^2 = \frac{4}{\pi}\int_0^{1/2\pi} r^2\sin^2\theta\,d\theta, \quad 1 = \frac{2}{\pi}\int_0^{1/2\pi} d\theta$$

$$r^2\ln r = \frac{4}{\pi}\int_0^{1/2\pi}\Big[r^2\sin^2\theta\ln 2r\sin\theta - \frac{1}{2}r^2\sin^2\theta\Big]d\theta \tag{3.4}$$

The validity of the last formula may be verified directly by integration and utilization of the definite integral

$$\int_0^{1/2\pi}\ln\sin\theta\,d\theta = -\frac{\pi}{2}\ln 2$$

We substitute (3.4) into the second integral (3.3) and introduce in the first a new variable by setting $t = r\sin\theta$. Then (3.3) reduces to the form

$$\int_0^{1/2\pi}\Big[\varphi(r\sin\theta) + \int_0^1 \varphi(t)R(r\sin\theta,\ t)\,dt\Big]d\theta = W_0 + V(r) \quad (r < 1) \tag{3.5}$$

Here

$$R(x,\ t) = \frac{2}{\pi}\int_0^\infty\Big\{\frac{\cos\lambda t}{\vartheta D\lambda^3}\Big[\cos\lambda x + C_1(\lambda)\frac{x^2}{2} + C_2(\lambda)\frac{x^2}{2}\Big(\ln 2x - \frac{3}{2}\Big) - 1\Big] -$$

$$- g(\lambda)\cos\lambda x\Big\}\,d\lambda \tag{3.6}$$

Equation (3.5) is the equation of Schlomilch [3]

$$\int_0^{1/2\pi} F(r\sin\theta)\,d\theta = f(r)$$

the unique solution of which with a continuous derivative will be

$$F(x) = \frac{2}{\pi}\Big[f(0) + x\int_0^{1/2\pi} f'(x\sin\theta)\,d\theta\Big] \tag{3.7}$$

In the original notation (3.7) takes the form

$$\varphi(x)+\int_0^1 \varphi(t)R(x,\ t)\,dt=\frac{2}{\pi}\left[W_0+x\int_0^{1/2\pi} V'(x\sin\theta)\,d\theta\right]\quad (0\leqslant x\leqslant 1)\qquad (3.8)$$

Equation (3.8) is a Fredholm integral equation of the second kind. If its solution with a continuous derivative in the interval [0,1] exists, then the formulas (3.1), (1.5) and (2.5) give the solution of the posed contact problem. Using Formula (3.2), one may obtain the expression for the pressure p in the contact region directly in terms of p

$$p=\frac{1}{\vartheta}\left[\frac{\varphi(1)}{\sqrt{1-r^2}}-\int_r^1 \frac{\varphi'(t)\,dt}{\sqrt{t^2-r^2}}\right]\qquad (3.9)$$

We will evaluate the force P acting on the layer from the side of the plate

$$P=2\pi\int_0^1 prdr=\frac{2\pi}{\vartheta}\int_0^1\left\{\frac{\varphi(1)}{\sqrt{1-r^2}}-\int_r^1\frac{\varphi'(t)\,dt}{\sqrt{t^2-r^2}}\right\}rdr=$$

$$=\frac{2\pi}{\vartheta}\left\{\varphi(1)-\int_0^1\left(\int_0^t\varphi'(t)\,\frac{rdr}{\sqrt{t^2-r^2}}\,dr\right)dt\right\}=\frac{2\pi}{\vartheta}\int_0^1\varphi(t)\,dt\qquad (3.10)$$

Thus, the solution of the posed problem has been determined for given W_0. In the case where W_0 is unknown but the magnitude of a concentrated force applied at the center is given, the equation expressing the equality of the forces, evaluated in (3.10), and the sum of the projections of all external forces applied to the plate on the z-axis serves for the determination of W_0.

The expression (3.6) may be simplified considerably.

4. We substitute in (3.6) the expression for C_1 from (2.6); as a result we obtain

$$R(x,\ t)=\frac{2}{\pi\vartheta D}\left\{I(x,\ t)+\frac{x^2}{2}\left[\frac{3+\nu}{2(1+\nu)}\int_0^\infty J_1(\lambda)\cos\lambda t\,dt-\right.\right.$$
$$\left.\left.-\frac{1-\nu}{1+\nu}\int_0^\infty\frac{\cos\lambda t}{\lambda}J_2(\lambda)\,d\lambda\right]-\frac{x^2}{2}\left(\ln 2x-\frac{1}{2}\right)\int_0^\infty J_1(\lambda)\cos\lambda t\,d\lambda\right\}-\qquad (4.1)$$
$$-\frac{1}{\pi}[G(x+t)+G(x-t)]$$

Here

$$I(x,\ t)=\int_0^\infty\frac{\cos\lambda t}{\lambda^3}\left(\cos\lambda x+\frac{\lambda^2 I_0(\lambda)}{2}x^2-1\right)d\lambda,\ G(x)=\int_0^\infty g(\lambda)\cos\lambda x\,d\lambda\qquad (4.2)$$

All integrals, except for $G(x)$, may be evaluated. We use the formulas [3]

$$\int_0^\infty \frac{\cos\lambda t}{\lambda} J_2(\lambda)\,d\lambda = \frac{1}{2}(1-2t^2), \qquad \int_0^\infty J_1(\lambda)\cos\lambda t\,d\lambda = \sqrt{1-t^2} \tag{4.3}$$

We now differentiate I with respect to x

$$\frac{\partial I}{\partial x} = \int_0^\infty \frac{\cos\lambda t}{\lambda^2}[-\sin\lambda x + \lambda x J_0(\lambda)]\,d\lambda \tag{4.4}$$

Differentiation is justified, since the obtained integral converges uniformly with respect to x. We introduce into the consideration the integral

$$\Pi = \int_0^\infty \frac{\cos\lambda t}{\lambda^2+\beta^2}[-\sin\lambda x + \lambda x J_0(\lambda)]\,d\lambda \qquad (\beta > 0)$$

Obviously its limiting value for $\beta \to 0$ coincides with $\partial I/\partial x$.

The following formulas hold [4]:

$$\int_0^\infty \frac{\cos\lambda t}{\lambda^2+\beta^2}\lambda J_0(\lambda)\,d\lambda = \operatorname{ch}\beta t K_0(\beta) \tag{4.5}$$

$$\int_0^\infty \frac{\sin\lambda x\cos\lambda t}{\lambda^2+\beta^2}\,d\lambda = \frac{1}{4\beta}\{e^{-\beta(x-t)}\operatorname{Ei}[\beta(x-t)] + e^{-\beta(x+t)}\operatorname{Ei}[\beta(x+t)] - $$
$$- e^{\beta(x-t)}\operatorname{Ei}[-\beta(x-t)] - e^{\beta(x+t)}\operatorname{Ei}[-\beta(x+t)]\} \tag{4.6}$$

Here $K_0(\beta)$ is the Macdonald function and $\operatorname{Ei}(\beta)$ is the exponential integral. Using their expansions for small β

$$K_0(\beta) = -\ln\beta + \ln 2 + C + \ldots \quad \operatorname{Ei}[\beta] = C + \ln|\beta| + \beta + \ldots$$

and Formulas (4.6) and (4.7) we obtain

$$\frac{\partial I}{\partial x} = \lim_{\beta\to 0}\Pi = x(2C + \ln 2 - 1) + \frac{x-t}{2}\ln|x-t| + \frac{x+t}{2}\ln(x+t) \tag{4.7}$$

Here C is Euler's constant ($C = 0.5772$). Integrating (4.7) with respect to x we obtain the expression for the unknown integral

$$I = \tfrac{1}{2}x^2(2C + \ln 2 - 1) + \tfrac{1}{4}(x-t)^2(\ln|x-t| - \tfrac{1}{2}) +$$
$$+ \tfrac{1}{4}(x+t)^2(\ln|x+t| - \tfrac{1}{2}) - \tfrac{1}{2}t^2(\ln t - \tfrac{1}{2}) \tag{4.8}$$

Substitution in (4.1) of the integrals (4.3), (4.8) gives

$$R(x, t) = \gamma K(x, t) - \frac{1}{\pi}[G(x+t) + G(x-t)]$$

Here

$$K(x, t) = x^2\left[2C + \ln 2 - \frac{3+\nu}{2(1+\nu)}(1-\sqrt{1-t^2}) + \frac{1-\nu}{1+\nu}t^2\right] + \\ + H(x-t) + H(x+t) - 2H(t) - \sqrt{1-t^2}\,\frac{H(2x)}{2}$$

$$H(x) = \tfrac{1}{2}x^2(\ln|x| - \tfrac{1}{2}), \qquad \gamma = \frac{1}{\pi\vartheta D}$$

In the particular case where one is concerned with the interaction of the plate with the half-space ($h \to \infty$), the function $g(\lambda)$, and with it $G(x)$, vanishes and the kernel $R(x, t)$ of Equation (3.8) assumes the specially simple form

$$R(x, t) = \gamma K(x, t)$$

The free term of Equation (3.8) in this case does not change. Then Equation (3.8) and the other formulas give a new solution of the earlier studied problem (cf, for example, [5,6]).

5. As and example we consider a plate, at the center of which there is applied a concentrated force P. In this case $V(r) \equiv 0$ and Equation (3.8) takes the form

$$\omega(x) + \int_0^1 R(x, t)\,\omega(t)\,dt = 1 \qquad (0 \leqslant x \leqslant 1) \tag{5.1}$$

into which there has been introduced for the convenience of the computations the new unknown $\omega(x)$, related to $\phi(x)$ by the equation

$$\varphi(x) = \frac{2}{\pi}W_0\omega(x) \tag{5.2}$$

In the case of contact of the plate with an elastic half-space the equation is

$$\omega(x) + \gamma\int_0^1 K(x, t)\,\omega(t)\,dt = 1 \qquad (0 \leqslant x \leqslant 1) \tag{5.3}$$

We achieve the solution of (5.3) by a numerical method, replacing its integral by an approximate expression based on the trapezoidal rule. Satisfying Equation (5.3) at a number of points we obtain a system of linear algebraic equations. Its solution for not-too-large values of the parameter γ has been found by the method of successive approximations. Using (3.10) and (5.2) we find the connection between the applied force P and the deflection of the plate under it

$$\alpha(\gamma) = \int_0^1 \omega(t)\,dt = \frac{P\vartheta}{4W_0}$$

The following table gives the results of the computed values of the function $\omega(x)$ of (5.3) for γ = 0.0, 0.3, 0.5, 1.0, 1.5.

TABLE

	$\gamma=0,0$	$\gamma=0.3$	$\gamma=0.5$	$\gamma=1.0$	$\gamma=1.5$
0.0	1.000	1.000	1.000	1.000	1.000
0.1	1.000	0.992	0.987	0.979	0.973
0.2	1.000	0.971	0.957	0.931	0.913
0.3	1.000	0.943	0.915	0.865	0.830
0.4	1.000	0.908	0.863	0.783	0.730
0.5	1.000	0.867	0.802	0.689	0.615
0.6	1.000	0.821	0.734	0.582	0.484
0.7	1.000	0.769	0.657	0,463	0.338
0.8	1.000	0.712	0.574	0.332	0.177
0.9	1.000	0.651	0.482	0.187	—0.002
1.0	1.000	0.584	0.382	0.029	—0.199

For the stated values of γ the values of the function $a(\gamma)$ in accordance with (5.4) are

$$\begin{array}{llllll} \gamma = 0.00 & 0.3 & 0.5 & 1.0 & 1.5 \\ \alpha(\gamma) = 1.000 & 0.843 & 0.766 & 0.633 & 0.546 \end{array}$$

It follows from the table that for $\gamma > 1.5$ the function $\omega(x)$ remains negative for x near unity. Then, in correspondence with (3.9), the pressure p likewise remains negative for $r = 1 (W_0 > 0)$. In the formulation of the problem complete attachment of the plate to the layer has been assumed, and therefore the obtained results are fully applicable.

If it is admitted that the plate lies freely on the layer, its edge will tear away from the layer for the stated values of γ and the above solution loses meaning. Clearly such tearing occurs for values of γ larger than γ_* for which the condition $\omega(1) = 0$ is fulfilled.

More detailed computations than those given in the table show the critical value $\gamma : \gamma_* = 1.053$. For the study of the problem of a layer of finite thickness the computations may be performed by the same method using the table of the function $G(x)$ of [1].

We present the results of computing γ_* as a function of the thickness h

$\frac{1}{h} = 0$	0.5	1	2
$\gamma_* = 1.053$	1.019	0.889	0.623

This dependence of $\gamma_*(h^{-1})$ permits the following conclusion to be drawn: for unchanging materials of the layer and plate the minimum thickness of the plate corresponding to contact over its entire area increases with decreasing thickness of the layer.

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